Modeling of Surface-Mounted PMSM in Different Frames of Reference

***Abstract**: The model of PMSM can be defined in natural three phase reference frame (ABC frame), in equivalent two phase frame (α-β frame) and in synchronously rotating reference frame (Q-D frame). The model is constructed from first principles in ABC frame and then suitable transformations are used to get the model in the two latter frames of reference. The transformations are dependent on the positions of the axes in each reference frame relative to each other and on the definition of the rotor position angle. Rotor position angle can have two different definitions (D-alignment and Q-alignment) and the type of alignment is a permanent feature of the motor after attaching its encoder. The purpose of this paper is to obtain the models under a variety of geometrical assumptions and hence to show how such different assumptions affect the obtained models.*

Key Words: PMSM model, PMSM α-β Model, PMSM d-q model, position encoder alignment

1. Introduction

The rotor position is the angle between the axis of a rotor magnetic axis and the magnetic axis of stator phase A. The rotor magnetic axis enclosing the angle can be the direct axis (d-axis) as in figure (1) or can be the q-axis as in figure (2) and figure (3). Any of these two different possibilities for defining the rotor position can be considered by the motors manufacturers. Therefore, we can find a motor whose position encoder is Q-aligned or a motor with D-aligned encoder. This can be identified for a give motor by turning its rotor and observing the resulting voltage waveforms with respect to the zero angle of the encoder. D-alignment is characterized with back EMF waveforms that are sine functions of encoder angle (i.e. zero crossing of the back EMF of one of the phases coincides with the zero of the encoder, this should be named phase A). Q-alignment is characterized with back EMF waveforms that are cosine functions of the encoder angle (i.e. the peak of the back EMF of one of the phases coincides with the zero of the encoder, this should be named phase A). The figures (1),(2) and (3) are representing a machine with one pair of poles merely for sake of simplicity, but the same concepts are applicable to machines with any number of pole pairs $\boldsymbol{P}$.

Of course the way in which rotor position is defined dictates certain type of trigonometric function to describe the permanent magnet flux linking the stator phases as will be shown in the following sections. Also, the phase relationship between the β-axis and the α-axis can be as shown in figure (2) and figure (3) where the β-axis leads α-axis as in figure (2) or can be that α-axis leads β-axis as in figure (3). Typically, α-axis is assumed to be in phase with the axis of phase A

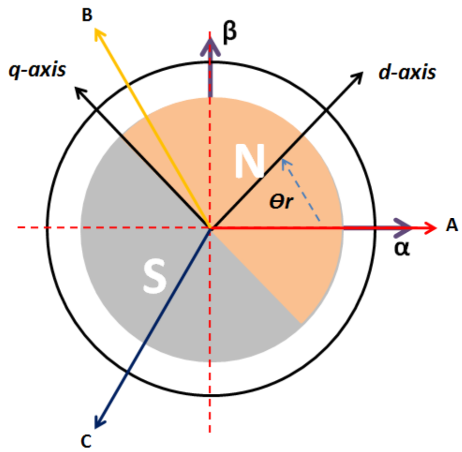


Figure (1). Rotor position is zero when the axis of the magnet's North Pole is aligned with the axis of phase A, β-axis is leading α-axis (D-alignment)

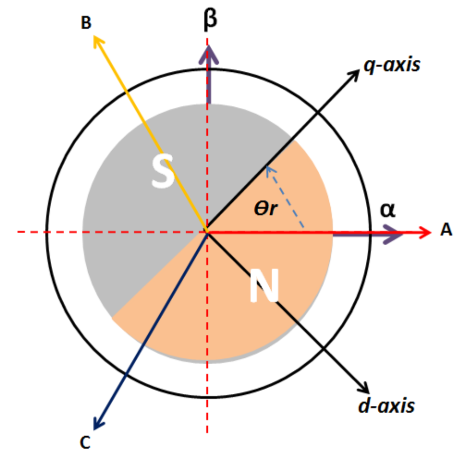


Figure (2). Rotor position is zero when the axis of the magnet's North Pole is perpendicular to the axis of phase A, β-axis is leading α-axis (Q-alignment)

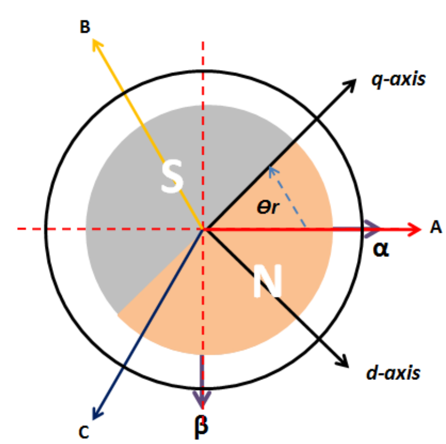


Figure (3). Rotor position is zero when the axis of the magnet's North Pole is perpendicular to the axis of phase A and β-axis is lagging behind α-axis (Q-alignment)

2. Transformations

1. Phase Transformation (Clarke's Transformation)

Clarke's transformation K_{3s}^{2s} is used to transform three phases' quantities F_{ABC} (voltages and currents) into two equivalent phases $F_{\alpha\beta}$ in the stationary reference frame of the stator, and its reverse K_{2s}^{3s} is to obtain *ABC* quantities from *α-β* quantities

$$F_{\alpha\beta} = K_{3s}^{2s}.F_{ABC} \quad , F_{ABC} = K_{2s}^{3s}.F_{\alpha\beta}$$

It can be deduced from the geometry of the *ABC* vectors with respect to the *α-β* axes that we will have three cases as follows:

i. <u>Case 1: β-axis is leading α-axis, D-alignmnet</u>

This is as shown in figure (1)

$$K_{3s}^{2s} = \sqrt{\frac{2}{3}}.\begin{bmatrix} 1 & -1/2 & -1/2 \\ 0 & \sqrt{3}/2 & -\sqrt{3}/2 \\ 1/\sqrt{2} & 1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix} \quad \text{.......(1)}$$

The square root of (2/3) is chosen to equate the power resulting from the voltage and currents across the two different frames of reference. From the properties of orthogonal transforms, the reverse transformation K_{2s}^{3s} is simply the transpose of K_{3s}^{2s}

$$K_{2s}^{3s} = {K_{3s}^{2s}}^T = \sqrt{\frac{2}{3}}.\begin{bmatrix} 1 & 0 & 1/\sqrt{2} \\ -1/2 & \sqrt{3}/2 & 1/\sqrt{2} \\ -1/2 & -\sqrt{3}/2 & 1/\sqrt{2} \end{bmatrix} \quad \text{.......(2)}$$

ii. <u>Case 2: β-axis is leading α-axis, Q-alignment</u>

This is as shown in figure (2) and it is exactly the same as in figure (1), therefore we may not repeat here equations (1) and (2)

iii. <u>Case 3: α-axis is leading β –axis, Q-alignment</u>

This is as shown in figure (3)

$$K_{3s}^{2s} = \sqrt{\frac{2}{3}}.\begin{bmatrix} 1 & -1/2 & -1/2 \\ 0 & -\sqrt{3}/2 & \sqrt{3}/2 \\ 1/\sqrt{2} & 1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix} \quad \text{.......(3)}$$

And the reverse transformation K_{2s}^{3s} is simply the transpose of K_{3s}^{2s} is:

$$K_{2s}^{3s} = {K_{3s}^{2s}}^T = \sqrt{\frac{2}{3}}.\begin{bmatrix} 1 & 0 & 1/\sqrt{2} \\ -1/2 & -\sqrt{3}/2 & 1/\sqrt{2} \\ -1/2 & \sqrt{3}/2 & 1/\sqrt{2} \end{bmatrix} \quad \text{.......(4)}$$

2. **Commutator transformation (Park's Transformation)**

This transformation is used to transform 2 phase quantities $F_{\alpha\beta}$ in the stationary reference frame α-β to their equivalent components F_{qd} in a rotating reference frame whose axes q, d are arranged such that an angle *Θr* is enclosed between the 2 set of axes and its reverse is used to obtain the *α-β* quantities from the q-d quantities

$$F_{qd} = K_{2s}^{2r}.F_{\alpha\beta}, \quad F_{\alpha\beta} = K_{2r}^{2s}.F_{qd}$$

The reverse transform is obtained by transposing the forward transform since they are orthogonal transformations. From the geometry of the figures (1), (2) and (3) it follows that we will have three different geometries, in which the rotor angle *Θr* is enclosed:

- either between d-axis and α-axis while β-axis is leading α-axis (case-1)
- or between q-axis and α-axis while β-axis is leading α-axis (case-2)
- or between q-axis and α-axis while α-axis is leading β -axis (case-3)

i. **<u>Case 1: β-axis is leading α-axis, D-alignment</u>**

As shown in figure (1), to transform from α-β quantities to q-d quantities, the transformation is:

$$K_{2s}^{2r} = \begin{bmatrix} -\sin(\theta_r) & \cos(\theta_r) \\ \cos(\theta_r) & \sin(\theta_r) \end{bmatrix} \quad \text{.....(5)}$$

And the reverse transformation used to transform from q-d quantities to α-β quantities is:

$$K_{2r}^{2s} = \begin{bmatrix} -\sin(\theta_r) & \cos(\theta_r) \\ \cos(\theta_r) & \sin(\theta_r) \end{bmatrix} \quad \text{.....(6)}$$

(Notice that the reverse is the same as the forward transform in this geometry)

ii. **<u>Case 2: β-axis is leading α-axis, Q-alignment</u>**

As shown in figure (2), to transform from α-β quantities to q-d quantities, the transformation is:

$$K_{2s}^{2r} = \begin{bmatrix} \cos(\theta_r) & \sin(\theta_r) \\ \sin(\theta_r) & -\cos(\theta_r) \end{bmatrix} \quad \text{.....(7)}$$

And the reverse transformation used to transform from q-d quantities to α-β quantities is:

$$K_{2r}^{2s} = \begin{bmatrix} \cos(\theta_r) & \sin(\theta_r) \\ \sin(\theta_r) & -\cos(\theta_r) \end{bmatrix} \quad \text{.....(8)}$$

(Again, notice that the inverse is the same as the forward transform in this geometry)

iii. **<u>Case 3: α -axis is leading β-axis, Q-alignment</u>**

As shown in figure (3), to transform from α-β quantities to q-d quantities, the transformation is:

$$K_{2s}^{2r} = \begin{bmatrix} \cos(\theta_r) & -\sin(\theta_r) \\ \sin(\theta_r) & \cos(\theta_r) \end{bmatrix} \quad \text{.....(9)}$$

And the reverse transformation used to transform from q-d quantities to α-β quantities is:

$$K_{2r}^{2s} = \begin{bmatrix} \cos(\theta_r) & \sin(\theta_r) \\ -\sin(\theta_r) & \cos(\theta_r) \end{bmatrix} \quad \text{.......(10)}$$

3. Combined Phase and Commutator Transformations

Using matrix algebra and trigonometric identities, the two successive transformations in the forward and reverse path can be found. In the following section the forward and the reverse combined transformation will be obtained.

i. Case 1: β-axis is leading α-axis, D-alignment

From equations (1) and (5) with omitting the third row since under normal operation the three phase system is balanced.

$$K_{3s}^{2r} = K_{2s}^{2r}.K_{3s}^{2s} = \begin{bmatrix} -\sin(\theta_r) & \cos(\theta_r) \\ \cos(\theta_r) & \sin(\theta_r) \end{bmatrix}.\sqrt{\frac{2}{3}}.\begin{bmatrix} 1 & -1/2 & -1/2 \\ 0 & \sqrt{3}/2 & -\sqrt{3}/2 \end{bmatrix}$$

$$K_{3s}^{2r} = \sqrt{\frac{2}{3}}.\begin{bmatrix} -\sin(\theta_r) & -\sin(\theta_r - 2\pi/3) & -\sin(\theta_r + 2\pi/3) \\ \cos(\theta_r) & \cos(\theta_r - 2\pi/3) & \cos(\theta_r + 2\pi/3) \end{bmatrix} \quad \text{.......(11)}$$

The inverse transform is simply given by the transpose as:

$$K_{2r}^{3s} = \sqrt{\frac{2}{3}}.\begin{bmatrix} -\sin(\theta_r) & \cos(\theta_r) \\ -\sin(\theta_r - 2\pi/3) & \cos(\theta_r - 2\pi/3) \\ -\sin(\theta_r + 2\pi/3) & \cos(\theta_r + 2\pi/3) \end{bmatrix} \quad \text{.......(12)}$$

ii. Case 2: β-axis is leading α-axis, Q-alignment

From equations (1) and (7):

$$K_{3s}^{2r} = K_{2s}^{2r}.K_{3s}^{2s} = \begin{bmatrix} \cos(\theta_r) & \sin(\theta_r) \\ \sin(\theta_r) & -\cos(\theta_r) \end{bmatrix}.\sqrt{\frac{2}{3}}.\begin{bmatrix} 1 & -1/2 & -1/2 \\ 0 & \sqrt{3}/2 & -\sqrt{3}/2 \end{bmatrix}$$

$$K_{3s}^{2r} = \sqrt{\frac{2}{3}}.\begin{bmatrix} \cos(\theta_r) & \cos(\theta_r - 2\pi/3) & \cos(\theta_r + 2\pi/3) \\ \sin(\theta_r) & \sin(\theta_r - 2\pi/3) & \sin(\theta_r + 2\pi/3) \end{bmatrix} \quad \text{.......(13)}$$

The inverse transform is simply given by the transpose as:

$$K_{2r}^{3s} = \sqrt{\frac{2}{3}}.\begin{bmatrix} \cos(\theta_r) & \sin(\theta_r) \\ \cos(\theta_r - 2\pi/3) & \sin(\theta_r - 2\pi/3) \\ \cos(\theta_r + 2\pi/3) & \sin(\theta_r + 2\pi/3) \end{bmatrix} \quad \text{.......(14)}$$

iii. **Case 3: α -axis is leading β-axis, Q-alignment**

From equations (3) and (9)

$$K_{3s}^{2r} = K_{2s}^{2r}.K_{3s}^{2s} = \begin{bmatrix} \cos(\theta_r) & -\sin(\theta_r) \\ \sin(\theta_r) & \cos(\theta_r) \end{bmatrix}.\sqrt{\frac{2}{3}}.\begin{bmatrix} 1 & -1/2 & -1/2 \\ 0 & -\sqrt{3}/2 & \sqrt{3}/2 \end{bmatrix}$$

$$K_{3s}^{2r} = \sqrt{\frac{2}{3}}.\begin{bmatrix} \cos(\theta_r) & \cos(\theta_r - 2\pi/3) & \cos(\theta_r + 2\pi/3) \\ \sin(\theta_r) & \sin(\theta_r - 2\pi/3) & \sin(\theta_r + 2\pi/3) \end{bmatrix} \quad \text{......(15)}$$

The inverse transform is simply given by the transpose as:

$$K_{2r}^{3s} = \sqrt{\frac{2}{3}}.\begin{bmatrix} \cos(\theta_r) & \sin(\theta_r) \\ \cos(\theta_r - 2\pi/3) & \sin(\theta_r - 2\pi/3) \\ \cos(\theta_r + 2\pi/3) & \sin(\theta_r + 2\pi/3) \end{bmatrix} \quad \text{......(16)}$$

3. PMSM Model in Natural Frame of Reference (ABC frame)

In Matrix form,

$$\mathbf{v}_{abcs} = \mathbf{r}_s \mathbf{i}_{abcs} + p\boldsymbol{\lambda}_{abcs} \quad \text{......(17)}$$

For phase voltages V_{abc}, stator currents i_{abc} , and flux linkages λ_{abc}, the following vector representation applies:

$$(\mathbf{f}_{abcs})^T = [f_{as} \quad f_{bs} \quad f_{cs}] \quad \text{......(18)}$$

and

$$\mathbf{r}_s = \begin{bmatrix} r_s & 0 & 0 \\ 0 & r_s & 0 \\ 0 & 0 & r_s \end{bmatrix} \quad \text{......(19)}$$

Where r_s is the resistance per phase of stator winding (windings are assumed to be exactly symmetrical) The flux linkages equations can be expressed in matrix form as follows:

$$\boldsymbol{\lambda}_{abcs} = \mathbf{L}_s \mathbf{i}_{abcs} + \lambda_m \quad \text{......(20)}$$

Where $\mathbf{L}_S$ is the inductance matrix and λ_m is the magnitude of the flux linkage established by the permanent magnet as viewed by the three stator phase windings. In other words, the magnitude of $p\lambda_m$ would be the magnitude of the open-circuit voltage induced in each stator phase winding when the rotor moves, hence λ_m' in the following equations (21) and (22) is the back EMF constant of the motor. Depending on the definition of the rotor position, if it follows the D-alignement as in figure (1), then the vector λ_m of equation (20) is given by equation (21).

$$\lambda_m = \begin{bmatrix} \lambda_{asm} \\ \lambda_{bsm} \\ \lambda_{csm} \end{bmatrix} = \lambda'_m \begin{bmatrix} \cos(\theta_r) \\ \cos(\theta_r - 2\pi/3) \\ \cos(\theta_r + 2\pi/3) \end{bmatrix} \qquad (21)$$

If the difinition of the rotor position follows the Q-alignement as in figure (2), and figure (3), then λ_m is given by equation (22).

$$\lambda_m = \begin{bmatrix} \lambda_{asm} \\ \lambda_{bsm} \\ \lambda_{csm} \end{bmatrix} = \lambda'_m \begin{bmatrix} \sin(\theta_r) \\ \sin(\theta_r - 2\pi/3) \\ \sin(\theta_r + 2\pi/3) \end{bmatrix} \qquad (22)$$

For surface-mounted magnets the air gap is uniform (relative permeability of permanent magnet materials are nearly the same of air). Therefore inductance matrix $\mathbf{L}_s$ is given by equation (23)

$$\mathbf{L}_s = \begin{bmatrix} L_{ls} + L_{ms} & -\frac{1}{2}L_{ms} & -\frac{1}{2}L_{ms} \\ -\frac{1}{2}L_{ms} & L_{ls} + L_{ms} & -\frac{1}{2}L_{ms} \\ -\frac{1}{2}L_{ms} & -\frac{1}{2}L_{ms} & L_{ls} + L_{ms} \end{bmatrix} \qquad (23)$$

where L_{ls} , L_{ms} represent the stator's leakage inductance and magnetizing inductance per phase.

From equations (21) and (22), the open circuit voltage or the back EMF E_{abc} is given by time derivative of λ_m. For D-alignment as of figure (1):

$$\begin{bmatrix} e_{as} \\ e_{bs} \\ e_{cs} \end{bmatrix} = -\lambda'_m \frac{d\theta_r}{dt} \begin{bmatrix} \sin(\theta_r) \\ \sin\left(\theta_r - \frac{2}{3}\pi\right) \\ \sin\left(\theta_r + \frac{2}{3}\pi\right) \end{bmatrix} \qquad (24)$$

And for Q-alignment as of figure (2), figure (3):

$$\begin{bmatrix} e_{as} \\ e_{bs} \\ e_{cs} \end{bmatrix} = \lambda'_m \frac{d\theta_r}{dt} \begin{bmatrix} \cos(\theta_r) \\ \cos\left(\theta_r - \frac{2}{3}\pi\right) \\ \cos\left(\theta_r + \frac{2}{3}\pi\right) \end{bmatrix} \qquad (25)$$

Assuming a linear magnetic circuit and conservative coupling filed, an infintesaml part of electrical enrgy of a machine with **P** pair of poles is completley converted into an equal infintesmal part of mechanical energy. This can be written as :

$$\left(i_{as} e_{as} + i_{bs} e_{bs} + i_{cs} e_{cs}\right).dt = \left(1/P\right).T_e.d\theta_r \qquad (26)$$

Where T_e is the torque developed by the motor. It follows that for a D-aligned representation as shown in figure (1), T_e is given by equation (27) where some simplification is applied to the sum of products of the currents and voltage.

$$T_e(i_{as}, i_{bs}, i_{cs}, \theta_r) = P\lambda'_m \left(-\tfrac{3}{2} i_{as} \sin(\theta_r) + \tfrac{\sqrt{3}}{2}\left(i_{bs} - i_{cs}\right)\cos(\theta_r)\right) \qquad (27)$$

And for Q-aligned representation as shown in figure (2) and figure (3), the torque is given by equation (28)

$$T_e(i_{as}, i_{bs}, i_{cs}, \theta_r) = P\lambda'_m \left(\tfrac{3}{2} i_{as}\cos(\theta_r) + \tfrac{\sqrt{3}}{2}(i_{bs} - i_{cs})\sin(\theta_r)\right) \qquad (28)$$

The mechanical subsystem is described by equation (29) as follows:

$$T_e = T_L + B\omega_{rm} + J\, p\omega_{rm} \qquad (29)$$

Where ω_{rm} is the motor mechanical speed ($\omega_{rm} = \omega_r / P$)

4. Model in 2 Phase Stationary Frame (α-β frame)

Using Proper set of transformation, we will proceed to find the model of the PMSM in the α- β frame

i. <u>Case 1: β-axis is leading α-axis, D-alignment</u>

Applying the phase transformation to the voltage equation (17)

$$\left[K_{3s}^{2s}\right]^{-1}\mathbf{v}_{\alpha\beta} = \mathbf{r}_s\left[K_{3s}^{2s}\right]^{-1}\mathbf{i}_{\alpha\beta} + p\left[K_{3s}^{2s}\right]^{-1}\boldsymbol{\lambda}_{\alpha\beta} \qquad (1\text{-}1)$$

Taking into account that $[K_{3s}^{2s}]^{-1}$ is constant and independent of time, it can be taken out of the derivative operator scope, then pre-multiplying both sides by K_{3s}^{2s} yields

$$\mathbf{v}_{\alpha\beta} = \left[K_{3s}^{2s}\right]\mathbf{r}_s\left[K_{3s}^{2s}\right]^{-1}\mathbf{i}_{\alpha\beta} + p\boldsymbol{\lambda}_{\alpha\beta} \qquad (1\text{-}2)$$

From equations (1), (2), (21) and (23):

$$\boldsymbol{\lambda}_{\alpha\beta} = \left[K_{3s}^{2s}\right]\mathbf{L}_s\left[K_{3s}^{2s}\right]^{-1}\mathbf{i}_{\alpha\beta} + \left[K_{3s}^{2s}\right]\lambda'_m \begin{bmatrix} \cos(\theta_r) \\ \cos(\theta_r - \frac{2}{3}\pi) \\ \cos(\theta_r + \frac{2}{3}\pi) \end{bmatrix} \qquad (1\text{-}3)$$

This yields:

$$\begin{bmatrix} \lambda_\alpha \\ \lambda_\beta \\ \lambda_0 \end{bmatrix} = \begin{bmatrix} L_{ls} + \frac{3}{2}L_{ms} & 0 & 0 \\ 0 & L_{ls} + \frac{3}{2}L_{ms} & 0 \\ 0 & 0 & L_{ls} \end{bmatrix} \begin{bmatrix} i_\alpha \\ i_\beta \\ i_0 \end{bmatrix} + \lambda_m \begin{bmatrix} \cos(\theta_r) \\ \sin(\theta_r) \\ 0 \end{bmatrix} \qquad (1\text{-}4)$$

Where: $\boldsymbol{\lambda_m} = (\sqrt{\tfrac{3}{2}})\boldsymbol{\lambda'_m}$ (1-5)

$$\left[K_{3s}^{2s}\right]\mathbf{r}_s\left[K_{3s}^{2s}\right]^{-1}\mathbf{i}_{\alpha\beta} = \begin{bmatrix} r_s & 0 & 0 \\ 0 & r_s & 0 \\ 0 & 0 & r_s \end{bmatrix} \begin{bmatrix} i_\alpha \\ i_\beta \\ i_0 \end{bmatrix} \qquad (1\text{-}6)$$

Substituting from equations (1-4) and (1-6) in equation (1-2):

$$\begin{bmatrix} v_\alpha \\ v_\beta \\ v_0 \end{bmatrix} = \begin{bmatrix} r_s & 0 & 0 \\ 0 & r_s & 0 \\ 0 & 0 & r_s \end{bmatrix} \begin{bmatrix} i_\alpha \\ i_\beta \\ i_0 \end{bmatrix} + \begin{bmatrix} L_{ls} + \frac{3}{2} L_{ms} & 0 & 0 \\ 0 & L_{ls} + \frac{3}{2} L_{ms} & 0 \\ 0 & 0 & L_{ls} \end{bmatrix} \begin{bmatrix} pi_\alpha \\ pi_\beta \\ pi_0 \end{bmatrix} + \lambda_m \frac{d\,\theta_r}{dt} \begin{bmatrix} -\sin(\theta_r) \\ \cos(\theta_r) \\ 0 \end{bmatrix} \text{.............(1-7)}$$

Since PMSM are mostly connected to 3 wire systems where the neutral is isolated, the sum of three phase currents is zero and hence the zero component of the two-phase reference frame can be neglected. To get the torque expression, we just substitute the two phase currents according to the Clarke's transformation rule in the torque equation (27)

$$T_e(i_\alpha, i_\beta, \theta_r) = P.\lambda_m.\left(i_\beta \cos(\theta_r) - i_\alpha \sin(\theta_r)\right) \text{ ....(1-8)}$$

For many purposes (e.g. numerical simulation) it is desired to rewrite these equations (1-7), (1-8) and (29) such that the derivatives are all in the left hand side as follows:

$$\begin{aligned} pi_\alpha &= \frac{1}{L_{ls} + \frac{3}{2} L_{ms}} \left(v_\alpha - r_s\, i_\alpha + \lambda_m\, \omega_r \sin(\theta_r)\right) \\ pi_\beta &= \frac{1}{L_{ls} + \frac{3}{2} L_{ms}} \left(v_\beta - r_s\, i_\beta - \lambda_m\, \omega_r \cos(\theta_r)\right) \\ p\omega_r &= \left(\frac{P^2}{J}\right)..\lambda_m \left(i_\beta \cos\theta_r - i_\alpha \sin\theta_r\right) - \left(\frac{P}{J}\right) T_L - \left(\frac{B}{J}\right)\omega_r \\ p\theta_r &= \omega_r \end{aligned} \text{ ....(1-9)}$$

Recall from equation (1-5) that $\lambda_m = (\sqrt{\frac{3}{2}})\lambda_m'$

Obviously, the system of equations is nonlinear with independent control inputs v_α, v_β. The load torque T_L is considered as disturbance. Both motor voltages and currents are measurable quantities in the stationary frame and their respective quadrature components are computed using Clark's transformation mentioned earlier in equation (1). The commanded control values for the three phase quantities in the stationary frame are also obtained from the quadrature components using inverse Clark's transformation given in equation (2).

ii. <u>Case 2: β-axis is leading α-axis, Q-alignment</u>

Applying the phase transformation to the voltage equation (17)

$$\left[K_{3s}^{2s}\right]^{-1} \mathbf{v}_{\alpha\beta} = \mathbf{r}_s \left[K_{3s}^{2s}\right]^{-1} \mathbf{i}_{\alpha\beta} + p\left[K_{3s}^{2s}\right]^{-1} \boldsymbol{\lambda}_{\alpha\beta} \text{..(2-1)}$$

Taking into account that $[K_{3s}^{2s}]^{-1}$ is constant and independent of time, it can be taken out of the derivative operator scope, then pre-multiplying both sides by K_{3s}^{2s} yields

$$\mathbf{v}_{\alpha\beta} = \left[K_{3s}^{2s}\right] \mathbf{r}_s \left[K_{3s}^{2s}\right]^{-1} \mathbf{i}_{\alpha\beta} + p\boldsymbol{\lambda}_{\alpha\beta} \text{ ....(2-2)}$$

From equations (1), (2) and (22)

$$\boldsymbol{\lambda}_{\alpha\beta} = \left[K_{3s}^{2s}\right]\mathbf{L}_s \left[K_{3s}^{2s}\right]^{-1}\mathbf{i}_{\alpha\beta} + \left[K_{3s}^{2s}\right]\lambda_m^{'} \begin{bmatrix} \sin(\theta_r) \\ \sin(\theta_r - 2\pi/3) \\ \sin(\theta_r + 2\pi/3) \end{bmatrix} \quad \text{……………………………………..…..…(2-3)}$$

This yields:

$$\begin{bmatrix} \lambda_\alpha \\ \lambda_\beta \\ \lambda_0 \end{bmatrix} = \begin{bmatrix} L_{ls} + \frac{3}{2}L_{ms} & 0 & 0 \\ 0 & L_{ls} + \frac{3}{2}L_{ms} & 0 \\ 0 & 0 & L_{ls} \end{bmatrix} \begin{bmatrix} i_\alpha \\ i_\beta \\ i_0 \end{bmatrix} + \lambda_m \begin{bmatrix} \sin(\theta_r) \\ -\cos(\theta_r) \\ 0 \end{bmatrix} \quad \text{………………………….………......…(2-4)}$$

Where: $\boldsymbol{\lambda_m} = (\sqrt{\frac{3}{2}})\boldsymbol{\lambda_m^{'}}$………………………………………………………………………………...(2-5)

$$\left[K_{3s}^{2s}\right]\mathbf{r}_s\left[K_{3s}^{2s}\right]^{-1}\mathbf{i}_{\alpha\beta} = \begin{bmatrix} r_s & 0 & 0 \\ 0 & r_s & 0 \\ 0 & 0 & r_s \end{bmatrix} \begin{bmatrix} i_\alpha \\ i_\beta \\ i_0 \end{bmatrix} \quad \text{…………………………………………………………..………(2-6)}$$

Substituting from equations (2-4) and (2-6) in equation (2-2):

$$\begin{bmatrix} v_\alpha \\ v_\beta \\ v_0 \end{bmatrix} = \begin{bmatrix} r_s & 0 & 0 \\ 0 & r_s & 0 \\ 0 & 0 & r_s \end{bmatrix} \begin{bmatrix} i_\alpha \\ i_\beta \\ i_0 \end{bmatrix} + \begin{bmatrix} L_{ls} + \frac{3}{2}L_{ms} & 0 & 0 \\ 0 & L_{ls} + \frac{3}{2}L_{ms} & 0 \\ 0 & 0 & L_{ls} \end{bmatrix} \begin{bmatrix} pi_\alpha \\ pi_\beta \\ pi_0 \end{bmatrix} + \lambda_m \frac{d\,\theta_r}{dt} \begin{bmatrix} \cos(\theta_r) \\ \sin(\theta_r) \\ 0 \end{bmatrix} \quad \text{…………....(2-7)}$$

Since PMSM are mostly connected to 3-wire systems where the neutral is isolated, the sum of three phase currents is zero and hence the zero component of the two-phase reference frame can be neglected. To get the torque expression, we just substitute the two phase currents according to Clarke's transformation rule in the torque quation (28)

$$T_e(i_\alpha, i_\beta, \theta_r) = P.\lambda_m.\left(i_\alpha \cos(\theta_r) + i_\beta \sin(\theta_r)\right) \quad \text{……………………..……………………......(2-8)}$$

For many purposes (e.g. numerical simulation) it is desired to express these equations (2-7), (2-8) and (29), such that the derivatives are all in the left hand side as follows:

$$\begin{aligned} pi_\alpha &= \frac{1}{L_{ls} + \frac{3}{2}L_{ms}}\left(v_\alpha - r_s\, i_\alpha - \lambda_m\, \omega_r\, \cos(\theta_r)\right) \\ pi_\beta &= \frac{1}{L_{ls} + \frac{3}{2}L_{ms}}\left(v_\beta - r_s\, i_\beta - \lambda_m\, \omega_r\, \sin(\theta_r)\right) \\ p\omega_r &= \left(\frac{.P^2}{J}\right).\lambda_m\left(i_\alpha \cos\theta_r + i_\beta \sin\theta_r\right) - \left(\frac{P}{J}\right)T_L - \left(\frac{B}{J}\right)\omega_r \\ p\theta_r &= \omega_r \end{aligned} \quad \text{….………………………………..…….(2-9)}$$

Recall from equation(1-5) that $\lambda_m = (\sqrt{\frac{3}{2}})\lambda_m^{'}$

Obviously, the system of equations is nonlinear with independent control inputs v_α, v_β. The load torque T_L is considered as disturbance. Both motor voltages and currents are measurable quantities in the stationary frame and their respective quadrature components are computed using Clarke's transformation mentioned earlier in equation (1). The commanded control values for the three phase quantities in the stationary frame are also obtained from the quadrature components using inverse Clark's transformation given in equation (2).

iii. <u>Case 3: α -axis is leading β-axis, Q-alignment</u>

Applying the phase transformation to the voltage equation (17)

$$\left[K_{3s}^{2s}\right]^{-1}\mathbf{v}_{\alpha\beta} = \mathbf{r}_s\left[K_{3s}^{2s}\right]^{-1}\mathbf{i}_{\alpha\beta} + p\left[K_{3s}^{2s}\right]^{-1}\boldsymbol{\lambda}_{\alpha\beta} \quad \text{(3-1)}$$

Taking into account that $[K_{3s}^{2s}]^{-1}$ is constant and independent of time, it can be taken out of the derivative operator scope, then pre-multiplying both sides by K_{3s}^{2s} yields

$$\mathbf{v}_{\alpha\beta} = \left[K_{3s}^{2s}\right]\mathbf{r}_s\left[K_{3s}^{2s}\right]^{-1}\mathbf{i}_{\alpha\beta} + p\boldsymbol{\lambda}_{\alpha\beta} \quad \text{(3-2)}$$

From equations (3), (4), and (22)

$$\boldsymbol{\lambda}_{\alpha\beta} = \left[K_{3s}^{2s}\right]\mathbf{L}_s\left[K_{3s}^{2s}\right]^{-1}\mathbf{i}_{\alpha\beta} + \left[K_{3s}^{2s}\right]\lambda_m^{'}\begin{bmatrix}\sin\theta_r \\ \sin(\theta_r - \frac{2}{3}\pi) \\ \sin(\theta_r + \frac{2}{3}\pi)\end{bmatrix} \quad \text{(3-3)}$$

This yields:

$$\begin{bmatrix}\lambda_\alpha \\ \lambda_\beta \\ \lambda_0\end{bmatrix} = \begin{bmatrix}L_{ls} + \frac{3}{2}L_{ms} & 0 & 0 \\ 0 & L_{ls} + \frac{3}{2}L_{ms} & 0 \\ 0 & 0 & L_{ls}\end{bmatrix}\begin{bmatrix}i_\alpha \\ i_\beta \\ i_0\end{bmatrix} + \lambda_m\begin{bmatrix}\sin\theta_r \\ \cos\theta_r \\ 0\end{bmatrix} \quad \text{(3-4)}$$

Where: $\boldsymbol{\lambda_m} = (\sqrt{\frac{3}{2}})\boldsymbol{\lambda_m^{'}}$ ……(3-5)

$$\left[K_{3s}^{2s}\right]\mathbf{r}_s\left[K_{3s}^{2s}\right]^{-1}\mathbf{i}_{\alpha\beta} = \begin{bmatrix}r_s & 0 & 0 \\ 0 & r_s & 0 \\ 0 & 0 & r_s\end{bmatrix}\begin{bmatrix}i_\alpha \\ i_\beta \\ i_0\end{bmatrix} \quad \text{(3-6)}$$

Substituting from equations (3-4) and (3-6) in equation (3-2) :

$$\begin{bmatrix} v_\alpha \\ v_\beta \\ v_0 \end{bmatrix} = \begin{bmatrix} r_s & 0 & 0 \\ 0 & r_s & 0 \\ 0 & 0 & r_s \end{bmatrix} \begin{bmatrix} i_\alpha \\ i_\beta \\ i_0 \end{bmatrix} + \begin{bmatrix} L_{ls} + \frac{3}{2} L_{ms} & 0 & 0 \\ 0 & L_{ls} + \frac{3}{2} L_{ms} & 0 \\ 0 & 0 & L_{ls} \end{bmatrix} \begin{bmatrix} pi_\alpha \\ pi_\beta \\ pi_0 \end{bmatrix} + \lambda_m \frac{d\,\theta_r}{dt} \begin{bmatrix} \cos\theta_r \\ -\sin\theta_r \\ 0 \end{bmatrix} \quad \text{............(3-7)}$$

Since PMSM are mostly connected to 3-wire systems where the neutral is isolated, the sum of three phase currents is zero and hence the zero component of the two-phase reference frame can be neglected. To get the torque expression, we just substitute the two phase currents according to the Clarke's transformation rule in the torque equation (28)

$$T_e(i_\alpha, i_\beta, \theta_r) = P.\lambda_m.\left(i_\alpha \cos\theta_r - i_\beta \sin\theta_r\right) \quad \text{...(3-8)}$$

For many purposes (e.g. numerical simulation) it is desired to rewrite these equations (3-7), (3-8) and (29) such that the derivatives are all in the left hand side as follows:

$$\begin{aligned} pi_\alpha &= \frac{1}{L_{ls} + \frac{3}{2} L_{ms}} \left(v_\alpha - r_s\, i_\alpha - \lambda_m\, \omega_r \cos\theta_r\right) \\ pi_\beta &= \frac{1}{L_{ls} + \frac{3}{2} L_{ms}} \left(v_\beta - r_s\, i_\beta + \lambda_m\, \omega_r \sin\theta_r\right) \\ p\omega_r &= \left(\frac{P^2}{J}\right).\lambda_m \left(i_\alpha \cos\theta_r - i_\beta \sin\theta_r\right) - \left(\frac{P}{J}\right) T_L - \left(\frac{B}{J}\right)\omega_r \\ p\theta_r &= \omega_r \end{aligned} \quad \text{...(3-9)}$$

Recall from equation(3-5) that $\lambda_m = (\sqrt{\frac{3}{2}})\lambda_m'$

Obviously, the system of equations is nonlinear with independent control inputs v_α, v_β . The load torque T_L is considered as disturbance. Both motor voltages and currents are measurable quantities in the stationary frame and their respective quadrature components are computed using Clarke's transformation mentioned earlier in equations (3). The commanded control values for the three phase quantities in the stationary frame are also obtained from the quadrature components using inverse Clark's transformation given in equation (4).

5. Model in 2 phase synchronously rotating reference frame (d-q frame)

As followed in α-β frame, The model in the d-q frame will be derived in the three cases.

i. Case 1: β-axis is leading α-axis, D-alignment

We can define the following:

$$L_{\alpha\beta} = L_{ls} + \tfrac{3}{2} L_{ms}\,,\ \omega_r = p\theta_r \quad \text{..(1-1)}$$

Equation (1-7) can be rewritten (considering $i_0 = 0$) as:

$$\begin{bmatrix} v_\alpha \\ v_\beta \end{bmatrix} = \begin{bmatrix} r_s & 0 \\ 0 & r_s \end{bmatrix} \begin{bmatrix} i_\alpha \\ i_\beta \end{bmatrix} + \begin{bmatrix} L_{\alpha\beta} & 0 \\ 0 & L_{\alpha\beta} \end{bmatrix} \begin{bmatrix} pi_\alpha \\ pi_\beta \end{bmatrix} + \lambda_m \ \omega_r \begin{bmatrix} -\sin(\theta_r) \\ \cos(\theta_r) \end{bmatrix} \text{......(1-2)}$$

Substituting for $V_{\alpha\beta}$ and $i_{\alpha\beta}$ we get:

$$\left[K_{2r}^{2s}\right] \begin{bmatrix} v_q \\ v_d \end{bmatrix} = \begin{bmatrix} r_s & 0 \\ 0 & r_s \end{bmatrix} . \left[K_{2r}^{2s}\right] \begin{bmatrix} i_q \\ i_d \end{bmatrix} + \begin{bmatrix} L_{\alpha\beta} & 0 \\ 0 & L_{\alpha\beta} \end{bmatrix} p\left\{ \left[K_{2r}^{2s}\right] \begin{bmatrix} i_q \\ i_d \end{bmatrix} \right\} + \lambda_m \ \omega_r \begin{bmatrix} -\sin(\theta_r) \\ \cos(\theta_r) \end{bmatrix} \text{......(1-3)}$$

Pre-multiplying both sides by $\left[K_{2s}^{2r}\right]$, we get:

$$\left[K_{2s}^{2r}\right]\left[K_{2r}^{2s}\right] \begin{bmatrix} v_q \\ v_d \end{bmatrix} = \left[K_{2s}^{2r}\right] \begin{bmatrix} r_s & 0 \\ 0 & r_s \end{bmatrix} . \left[K_{2r}^{2s}\right] \begin{bmatrix} i_q \\ i_d \end{bmatrix} + \left[K_{2s}^{2r}\right] \begin{bmatrix} L_{\alpha\beta} & 0 \\ 0 & L_{\alpha\beta} \end{bmatrix} . p\left\{ \left[K_{2r}^{2s}\right] \begin{bmatrix} i_q \\ i_d \end{bmatrix} \right\} + \lambda_m \ \omega_r \left[K_{2s}^{2r}\right] \begin{bmatrix} -\sin(\theta_r) \\ \cos(\theta_r) \end{bmatrix}$$

(1-4)

Using equations (5) and (6) for Park's transform and through proper algebraic manipulation, the following can be reached:

$$\left[K_{2s}^{2r}\right] \begin{bmatrix} r_s & 0 \\ 0 & r_s \end{bmatrix} . \left[K_{2r}^{2s}\right] = \begin{bmatrix} r_s & 0 \\ 0 & r_s \end{bmatrix} \text{......(1-5)}$$

$$\lambda_m \ \omega_r \left[K_{2s}^{2r}\right] \begin{bmatrix} -\sin(\theta_r) \\ \cos(\theta_r) \end{bmatrix} = \lambda_m \ \omega_r \begin{bmatrix} 1 \\ 0 \end{bmatrix} \text{......(1-6)}$$

$$p\left\{ \left[K_{2r}^{2s}\right] \begin{bmatrix} i_q \\ i_d \end{bmatrix} \right\} = \left[K_{2r}^{2s}\right] \begin{bmatrix} pi_q \\ pi_d \end{bmatrix} + p\left\{ \left[K_{2r}^{2s}\right] \right\} \begin{bmatrix} i_q \\ i_d \end{bmatrix} \text{......(1-7)}$$

Then :

$$\left[K_{2s}^{2r}\right] \begin{bmatrix} L_{\alpha\beta} & 0 \\ 0 & L_{\alpha\beta} \end{bmatrix} . p\left\{ \left[K_{2r}^{2s}\right] \begin{bmatrix} i_q \\ i_d \end{bmatrix} \right\} = \left[K_{2s}^{2r}\right] \begin{bmatrix} L_{\alpha\beta} & 0 \\ 0 & L_{\alpha\beta} \end{bmatrix} . \left[K_{2r}^{2s}\right] \begin{bmatrix} pi_q \\ pi_d \end{bmatrix} + \left[K_{2s}^{2r}\right] \begin{bmatrix} L_{\alpha\beta} & 0 \\ 0 & L_{\alpha\beta} \end{bmatrix} p\left\{ \left[K_{2r}^{2s}\right] \right\} \begin{bmatrix} i_q \\ i_d \end{bmatrix}$$

(1-8)

$$\left[K_{2s}^{2r}\right] \begin{bmatrix} L_{\alpha\beta} & 0 \\ 0 & L_{\alpha\beta} \end{bmatrix} \left[K_{2r}^{2s}\right] = \begin{bmatrix} L_{\alpha\beta} & 0 \\ 0 & L_{\alpha\beta} \end{bmatrix} \text{......(1-9)}$$

$$\left[K_{2s}^{2r}\right] \begin{bmatrix} L_{\alpha\beta} & 0 \\ 0 & L_{\alpha\beta} \end{bmatrix} . p\left\{ \left[K_{2r}^{2s}\right] \begin{bmatrix} i_q \\ i_d \end{bmatrix} \right\} = \begin{bmatrix} L_{\alpha\beta} & 0 \\ 0 & L_{\alpha\beta} \end{bmatrix} . \begin{bmatrix} pi_q \\ pi_d \end{bmatrix} + \begin{bmatrix} 0 & \omega_r L_{\alpha\beta} \\ -\omega_r L_{\alpha\beta} & 0 \end{bmatrix} \begin{bmatrix} i_q \\ i_d \end{bmatrix} \text{......(1-10)}$$

Therefore, equation (1-4) can be rewritten as:

$$\begin{bmatrix} v_q \\ v_d \end{bmatrix} = \begin{bmatrix} r_s & \omega_r L_{\alpha\beta} \\ -\omega_r L_{\alpha\beta} & r_s \end{bmatrix} . \begin{bmatrix} i_q \\ i_d \end{bmatrix} + \begin{bmatrix} L_{\alpha\beta} & 0 \\ 0 & L_{\alpha\beta} \end{bmatrix} . \begin{bmatrix} pi_q \\ pi_d \end{bmatrix} + \lambda_m \ \omega_r \begin{bmatrix} 1 \\ 0 \end{bmatrix} \text{......(1-11)}$$

Also the torque equation in α-β frame (1-8) can be written as:

$$T_e(i_\alpha, i_\beta, \theta_r) = P.\lambda_m.\left[-\sin(\theta_r)\ \ \cos(\theta_r)\right]\begin{bmatrix} i_\alpha \\ i_\beta \end{bmatrix} \text{.......(1-12)}$$

Substituting in terms of q-d currents:

$$T_e(i_q, i_d) = P.\lambda_m.\left[-\sin(\theta_r)\ \ \cos(\theta_r)\right]\left[K_{2r}^{2s}\right]\begin{bmatrix} i_q \\ i_d \end{bmatrix} \text{.......(1-13)}$$

After proper algebraic manipulation using equation (6) we get:

$$T_e(i_q) = P.\lambda_m.i_q \text{.......(1-14)}$$

Recall that: $\lambda_m = (\sqrt{\frac{3}{2}})\lambda_m'$

Note that the applicable transformations in this case are given by equations (5), (6), (11) and (12).

ii. <u>Case 2: β-axis is leading α-axis, Q-alignment</u>

We can define the following:

$$L_{\alpha\beta} = L_{ls} + \tfrac{3}{2}L_{ms},\ \omega_r = p\theta_r \text{.......(2-1)}$$

Equation (3-7) can be rewritten (considering $i_0 = 0$) :

$$\begin{bmatrix} v_\alpha \\ v_\beta \end{bmatrix} = \begin{bmatrix} r_s & 0 \\ 0 & r_s \end{bmatrix}\begin{bmatrix} i_\alpha \\ i_\beta \end{bmatrix} + \begin{bmatrix} L_{\alpha\beta} & 0 \\ 0 & L_{\alpha\beta} \end{bmatrix}\begin{bmatrix} pi_\alpha \\ pi_\beta \end{bmatrix} + \lambda_m\ \omega_r\begin{bmatrix} \cos(\theta_r) \\ \sin(\theta_r) \end{bmatrix} \text{.......(2-2)}$$

Substituting for $V_{\alpha\beta}$ and $i_{\alpha\beta}$ we get:

$$\left[K_{2r}^{2s}\right]\begin{bmatrix} v_q \\ v_d \end{bmatrix} = \begin{bmatrix} r_s & 0 \\ 0 & r_s \end{bmatrix}.\left[K_{2r}^{2s}\right]\begin{bmatrix} i_q \\ i_d \end{bmatrix} + \begin{bmatrix} L_{\alpha\beta} & 0 \\ 0 & L_{\alpha\beta} \end{bmatrix}p\left\{\left[K_{2r}^{2s}\right]\begin{bmatrix} i_q \\ i_d \end{bmatrix}\right\} + \lambda_m\ \omega_r\begin{bmatrix} \cos(\theta_r) \\ \sin(\theta_r) \end{bmatrix} \text{.......(2-3)}$$

Pre-multiplying both sides by $\left[K_{2s}^{2r}\right]$, we get:

$$\left[K_{2s}^{2r}\right]\left[K_{2r}^{2s}\right]\begin{bmatrix} v_q \\ v_d \end{bmatrix} = \left[K_{2s}^{2r}\right]\begin{bmatrix} r_s & 0 \\ 0 & r_s \end{bmatrix}.\left[K_{2r}^{2s}\right]\begin{bmatrix} i_q \\ i_d \end{bmatrix} + \left[K_{2s}^{2r}\right]\begin{bmatrix} L_{\alpha\beta} & 0 \\ 0 & L_{\alpha\beta} \end{bmatrix}.p\left\{\left[K_{2r}^{2s}\right]\begin{bmatrix} i_q \\ i_d \end{bmatrix}\right\} + \lambda_m\ \omega_r\left[K_{2s}^{2r}\right]\begin{bmatrix} \cos(\theta_r) \\ \sin(\theta_r) \end{bmatrix}$$

(2-4)

Using equations (7) and (8) for Park's transform and through proper algebraic manipulation, the following can be reached:

$$\left[K_{2s}^{2r}\right]\begin{bmatrix} r_s & 0 \\ 0 & r_s \end{bmatrix}.\left[K_{2r}^{2s}\right] = \begin{bmatrix} r_s & 0 \\ 0 & r_s \end{bmatrix} \text{.......(2-5)}$$

$$\lambda_m\ \omega_r\left[K_{2s}^{2r}\right]\begin{bmatrix} \cos\theta_r \\ \sin\theta_r \end{bmatrix} = \lambda_m\ \omega_r\begin{bmatrix} 1 \\ 0 \end{bmatrix} \text{.......(2-6)}$$

$$p\left\{\left[K_{2r}^{2s}\right]\begin{bmatrix} i_q \\ i_d \end{bmatrix}\right\} = \left[K_{2r}^{2s}\right]\begin{bmatrix} pi_q \\ pi_d \end{bmatrix} + p\left\{\left[K_{2r}^{2s}\right]\right\}\begin{bmatrix} i_q \\ i_d \end{bmatrix} \text{.......(2-7)}$$

Then:

$$[K_{2s}^{2r}]\begin{bmatrix} L_{\alpha\beta} & 0 \\ 0 & L_{\alpha\beta} \end{bmatrix}.p\left\{[K_{2r}^{2s}]\begin{bmatrix} i_q \\ i_d \end{bmatrix}\right\} = [K_{2s}^{2r}]\begin{bmatrix} L_{\alpha\beta} & 0 \\ 0 & L_{\alpha\beta} \end{bmatrix}[K_{2r}^{2s}]\begin{bmatrix} pi_q \\ pi_d \end{bmatrix} + [K_{2s}^{2r}]\begin{bmatrix} L_{\alpha\beta} & 0 \\ 0 & L_{\alpha\beta} \end{bmatrix} p\{[K_{2r}^{2s}]\}\begin{bmatrix} i_q \\ i_d \end{bmatrix}$$

(2-8)

$$[K_{2s}^{2r}]\begin{bmatrix} L_{\alpha\beta} & 0 \\ 0 & L_{\alpha\beta} \end{bmatrix}[K_{2r}^{2s}] = \begin{bmatrix} L_{\alpha\beta} & 0 \\ 0 & L_{\alpha\beta} \end{bmatrix} \text{......(2-9)}$$

$$[K_{2s}^{2r}]\begin{bmatrix} L_{\alpha\beta} & 0 \\ 0 & L_{\alpha\beta} \end{bmatrix}.p\left\{[K_{2r}^{2s}]\begin{bmatrix} i_q \\ i_d \end{bmatrix}\right\} = \begin{bmatrix} L_{\alpha\beta} & 0 \\ 0 & L_{\alpha\beta} \end{bmatrix}.\begin{bmatrix} pi_q \\ pi_d \end{bmatrix} + \begin{bmatrix} 0 & \omega_r L_{\alpha\beta} \\ -\omega_r L_{\alpha\beta} & 0 \end{bmatrix}\begin{bmatrix} i_q \\ i_d \end{bmatrix} \text{......(2-10)}$$

Therefore, equation (2-4) can be rewritten as:

$$\begin{bmatrix} v_q \\ v_d \end{bmatrix} = \begin{bmatrix} r_s & \omega_r L_{\alpha\beta} \\ -\omega_r L_{\alpha\beta} & r_s \end{bmatrix}.\begin{bmatrix} i_q \\ i_d \end{bmatrix} + \begin{bmatrix} L_{\alpha\beta} & 0 \\ 0 & L_{\alpha\beta} \end{bmatrix}.\begin{bmatrix} pi_q \\ pi_d \end{bmatrix} + \lambda_m \, \omega_r \begin{bmatrix} 1 \\ 0 \end{bmatrix} \text{......(2-11)}$$

Also the torque equation in the α-β frame (2-8) can be written as:

$$T_e(i_\alpha, i_\beta, \theta_r) = P.\lambda_m.[\cos(\theta_r) \;\; \sin(\theta_r)]\begin{bmatrix} i_\alpha \\ i_\beta \end{bmatrix} \text{......(2-12)}$$

Substituting α-β currents by their q-d counter equivalents:

$$T_e(i_q, i_d) = P.\lambda_m.[\cos(\theta_r) \;\; \sin(\theta_r)].[K_{2r}^{2s}]\begin{bmatrix} i_q \\ i_d \end{bmatrix} \text{......(2-13)}$$

After proper algebraic manipulation using equation (8) we get:

$$T_e(i_q) = P.\lambda_m.i_q \text{......(2-14)}$$

Recall that: $\lambda_m = (\sqrt{\tfrac{3}{2}})\lambda_m'$

Note that the applicable transformations in this case are given by equations (7), (8), (13) and (14)

iii. <u>Case 3: α -axis is leading β-axis, Q-alignment</u>

We can define the following:

$$L_{\alpha\beta} = L_{ls} + \tfrac{3}{2} L_{ms}\,,\; \omega_r = p\theta_r \text{......(3-1)}$$

Equation (3-7) can be rewritten (considering $i_0 = 0$)

$$\begin{bmatrix} v_\alpha \\ v_\beta \end{bmatrix} = \begin{bmatrix} r_s & 0 \\ 0 & r_s \end{bmatrix}\begin{bmatrix} i_\alpha \\ i_\beta \end{bmatrix} + \begin{bmatrix} L_{\alpha\beta} & 0 \\ 0 & L_{\alpha\beta} \end{bmatrix}\begin{bmatrix} pi_\alpha \\ pi_\beta \end{bmatrix} + \lambda_m \, \omega_r \begin{bmatrix} \cos\theta_r \\ -\sin\theta_r \end{bmatrix} \text{......(3-2)}$$

Substituting for $V_{\alpha\beta}$ and $i_{\alpha\beta}$ we get

$$\left[K_{2r}^{2s}\right]\begin{bmatrix} v_q \\ v_d \end{bmatrix} = \begin{bmatrix} r_s & 0 \\ 0 & r_s \end{bmatrix}.\left[K_{2r}^{2s}\right]\begin{bmatrix} i_q \\ i_d \end{bmatrix} + \begin{bmatrix} L_{\alpha\beta} & 0 \\ 0 & L_{\alpha\beta} \end{bmatrix} p\left\{\left[K_{2r}^{2s}\right]\begin{bmatrix} i_q \\ i_d \end{bmatrix}\right\} + \lambda_m\ \omega_r \begin{bmatrix} \cos\theta_r \\ -\sin\theta_r \end{bmatrix} \quad \text{(3-3)}$$

Pre-multiplying both sides by $\left[K_{2s}^{2r}\right]$, we get:

$$\left[K_{2s}^{2r}\right]\left[K_{2r}^{2s}\right]\begin{bmatrix} v_q \\ v_d \end{bmatrix} = \left[K_{2s}^{2r}\right]\begin{bmatrix} r_s & 0 \\ 0 & r_s \end{bmatrix}.\left[K_{2r}^{2s}\right]\begin{bmatrix} i_q \\ i_d \end{bmatrix} + \left[K_{2s}^{2r}\right]\begin{bmatrix} L_{\alpha\beta} & 0 \\ 0 & L_{\alpha\beta} \end{bmatrix}.p\left\{\left[K_{2r}^{2s}\right]\begin{bmatrix} i_q \\ i_d \end{bmatrix}\right\} + \lambda_m\ \omega_r \left[K_{2s}^{2r}\right]\begin{bmatrix} \cos\theta_r \\ -\sin\theta_r \end{bmatrix} \quad \text{(3-4)}$$

Using equations (9) and (10) for park transforms and through proper algebraic manipulation, the following can be reached:

$$\left[K_{2s}^{2r}\right]\begin{bmatrix} r_s & 0 \\ 0 & r_s \end{bmatrix}.\left[K_{2r}^{2s}\right] = \begin{bmatrix} r_s & 0 \\ 0 & r_s \end{bmatrix} \quad \text{(3-5)}$$

$$\lambda_m\ \omega_r \left[K_{2s}^{2r}\right]\begin{bmatrix} \cos\theta_r \\ -\sin\theta_r \end{bmatrix} = \lambda_m\ \omega \begin{bmatrix} 1 \\ 0 \end{bmatrix} \quad \text{(3-6)}$$

$$p\left\{\left[K_{2r}^{2s}\right]\begin{bmatrix} i_q \\ i_d \end{bmatrix}\right\} = \left[K_{2r}^{2s}\right]\begin{bmatrix} pi_q \\ pi_d \end{bmatrix} + p\left\{\left[K_{2r}^{2s}\right]\right\}\begin{bmatrix} i_q \\ i_d \end{bmatrix} \quad \text{(3-7)}$$

Then:

$$\left[K_{2s}^{2r}\right]\begin{bmatrix} L_{\alpha\beta} & 0 \\ 0 & L_{\alpha\beta} \end{bmatrix}.p\left\{\left[K_{2r}^{2s}\right]\begin{bmatrix} i_q \\ i_d \end{bmatrix}\right\} = \left[K_{2s}^{2r}\right]\begin{bmatrix} L_{\alpha\beta} & 0 \\ 0 & L_{\alpha\beta} \end{bmatrix}\left[K_{2r}^{2s}\right]\begin{bmatrix} pi_q \\ pi_d \end{bmatrix} + \left[K_{2s}^{2r}\right]\begin{bmatrix} L_{\alpha\beta} & 0 \\ 0 & L_{\alpha\beta} \end{bmatrix} p\left\{\left[K_{2r}^{2s}\right]\right\}\begin{bmatrix} i_q \\ i_d \end{bmatrix} \quad \text{(3-8)}$$

$$\left[K_{2s}^{2r}\right]\begin{bmatrix} L_{\alpha\beta} & 0 \\ 0 & L_{\alpha\beta} \end{bmatrix}\left[K_{2r}^{2s}\right] = \begin{bmatrix} L_{\alpha\beta} & 0 \\ 0 & L_{\alpha\beta} \end{bmatrix} \quad \text{(3-9)}$$

$$\left[K_{2s}^{2r}\right]\begin{bmatrix} L_{\alpha\beta} & 0 \\ 0 & L_{\alpha\beta} \end{bmatrix}.p\left\{\left[K_{2r}^{2s}\right]\begin{bmatrix} i_q \\ i_d \end{bmatrix}\right\} = \begin{bmatrix} L_{\alpha\beta} & 0 \\ 0 & L_{\alpha\beta} \end{bmatrix}.\begin{bmatrix} pi_q \\ pi_d \end{bmatrix} + \begin{bmatrix} 0 & \omega_r L_{\alpha\beta} \\ -\omega_r L_{\alpha\beta} & 0 \end{bmatrix}\begin{bmatrix} i_q \\ i_d \end{bmatrix} \quad \text{(3-10)}$$

Therefore, equation (3-4) can be rewritten as:

$$\begin{bmatrix} v_q \\ v_d \end{bmatrix} = \begin{bmatrix} r_s & \omega_r L_{\alpha\beta} \\ -\omega_r L_{\alpha\beta} & r_s \end{bmatrix}.\begin{bmatrix} i_q \\ i_d \end{bmatrix} + \begin{bmatrix} L_{\alpha\beta} & 0 \\ 0 & L_{\alpha\beta} \end{bmatrix}.\begin{bmatrix} pi_q \\ pi_d \end{bmatrix} + \lambda_m\ \omega_r \begin{bmatrix} 1 \\ 0_r \end{bmatrix} \quad \text{(3-11)}$$

Also the torque equation (3-2) can be written as:

$$T_e(i_\alpha, i_\beta, \theta_r) = P.\lambda_m.\left[\cos(\theta_r)\ \ -\sin(\theta_r)\right]\begin{bmatrix} i_\alpha \\ i_\beta \end{bmatrix} \quad \text{(3-12)}$$

Substituting in terms of q-d currents:

$$T_e(i_q, i_d) = P.\lambda_m.\left[\cos(\theta_r)\ \ -\sin(\theta_r)\right]\left[K_{2r}^{2s}\right]\begin{bmatrix} i_q \\ i_d \end{bmatrix} \quad \text{(3-13)}$$

After proper algebraic manipulation using equation (10) we get:

$T_e(i_q) = P.\lambda_m.i_q$ ……(3-14)

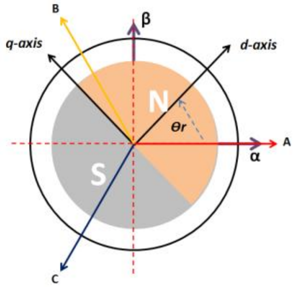
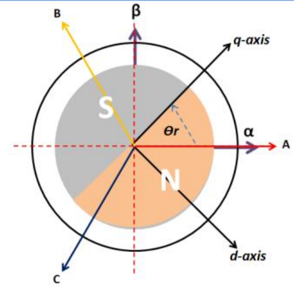
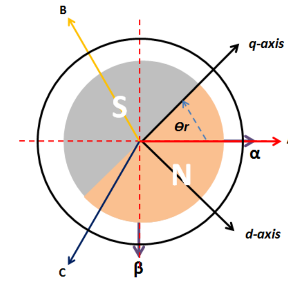
Recall that: $\lambda_m = (\sqrt{\frac{3}{2}})\lambda_m'$

Note that the applicable transformations in this case are given by equations (9), (10), (15) and (16).

6. Conclusion

It has been shown that different assumptions lead to differences in the equations of voltages and torque in α-β model, and that it leads to no difference in the equations of the q-d model. It is therefore very important to maintain consistency of assumptions by using the same set of transformations previously used for obtaining the model in implementing control within simulation or real time application. In particular, these assumptions should match the type of alignment (D-alignment / Q-alignment) adopted by the motor manufacturer for fixing the position sensor.

Appendix: 1

Case 1: D-Alignment, β is leading	Case 2: Q-Alignment, β is leading	Case 3: Q-Alignment, β is lagging
		
Power Conserving Phase Transformation : three stationary into two stationary (Clark)		
$K_{3s}^{2s}=\sqrt{\frac{2}{3}}\cdot\begin{bmatrix}1 & -1/2 & -1/2\\ 0 & \sqrt{3}/2 & -\sqrt{3}/2\\ 1/\sqrt{2} & 1/\sqrt{2} & 1/\sqrt{2}\end{bmatrix}$	$K_{3s}^{2s}=\sqrt{\frac{2}{3}}\cdot\begin{bmatrix}1 & -1/2 & -1/2\\ 0 & \sqrt{3}/2 & -\sqrt{3}/2\\ 1/\sqrt{2} & 1/\sqrt{2} & 1/\sqrt{2}\end{bmatrix}$	$K_{3s}^{2s}=\sqrt{\frac{2}{3}}\cdot\begin{bmatrix}1 & -1/2 & -1/2\\ 0 & -\sqrt{3}/2 & \sqrt{3}/2\\ 1/\sqrt{2} & 1/\sqrt{2} & 1/\sqrt{2}\end{bmatrix}$
Power Conserving Inverse Phase Transformation : two stationary into three stationary (Inverse Clark)		
$K_{2s}^{3s}=K_{3s}^{2s\,T}=\sqrt{\frac{2}{3}}\cdot\begin{bmatrix}1 & 0 & 1/\sqrt{2}\\ -1/2 & \sqrt{3}/2 & 1/\sqrt{2}\\ -1/2 & -\sqrt{3}/2 & 1/\sqrt{2}\end{bmatrix}$	$K_{2s}^{3s}=K_{3s}^{2s\,T}=\sqrt{\frac{2}{3}}\cdot\begin{bmatrix}1 & 0 & 1/\sqrt{2}\\ -1/2 & \sqrt{3}/2 & 1/\sqrt{2}\\ -1/2 & -\sqrt{3}/2 & 1/\sqrt{2}\end{bmatrix}$	$K_{2s}^{3s}=K_{3s}^{2s\,T}=\sqrt{\frac{2}{3}}\cdot\begin{bmatrix}1 & 0 & 1/\sqrt{2}\\ -1/2 & -\sqrt{3}/2 & 1/\sqrt{2}\\ -1/2 & \sqrt{3}/2 & 1/\sqrt{2}\end{bmatrix}$
Commutator Transform : two stationary phases into two synchronously rotating phases (Park)		
$K_{2s}^{2r}=\begin{bmatrix}-\sin(\theta_r) & \cos(\theta_r)\\ \cos(\theta_r) & \sin(\theta_r)\end{bmatrix}$	$K_{2s}^{2r}=\begin{bmatrix}\cos(\theta_r) & \sin(\theta_r)\\ \sin(\theta_r) & -\cos(\theta_r)\end{bmatrix}$	$K_{2s}^{2r}=\begin{bmatrix}\cos(\theta_r) & -\sin(\theta_r)\\ \sin(\theta_r) & \cos(\theta_r)\end{bmatrix}$
Inverse Commutator Transform : two synchronously rotating phases into two stationary phases (Inverse Park)		
$K_{2r}^{2s}=\begin{bmatrix}-\sin(\theta_r) & \cos(\theta_r)\\ \cos(\theta_r) & \sin(\theta_r)\end{bmatrix}$	$K_{2r}^{2s}=\begin{bmatrix}\cos(\theta_r) & \sin(\theta_r)\\ \sin(\theta_r) & -\cos(\theta_r)\end{bmatrix}$	$K_{2r}^{2s}=\begin{bmatrix}\cos(\theta_r) & \sin(\theta_r)\\ -\sin(\theta_r) & \cos(\theta_r)\end{bmatrix}$
Combined (3 stationary into 2 rotating) Transform		
$K_{3s}^{2r}=\sqrt{\frac{2}{3}}\cdot\begin{bmatrix}-\sin(\theta_r) & -\sin(\theta_r-2\pi/3) & -\sin(\theta_r+2\pi/3)\\ \cos(\theta_r) & \cos(\theta_r-2\pi/3) & \cos(\theta_r+2\pi/3)\end{bmatrix}$	$K_{3s}^{2r}=\sqrt{\frac{2}{3}}\cdot\begin{bmatrix}\cos(\theta_r) & \cos(\theta_r-2\pi/3) & \cos(\theta_r+2\pi/3)\\ \sin(\theta_r) & \sin(\theta_r-2\pi/3) & \sin(\theta_r+2\pi/3)\end{bmatrix}$	
Inverse Combined (2 rotating into 3 stationary) Transform		
$K_{2r}^{3s}=\sqrt{\frac{2}{3}}\cdot\begin{bmatrix}-\sin(\theta_r) & \cos(\theta_r)\\ -\sin(\theta_r-2\pi/3) & \cos(\theta_r-2\pi/3)\\ -\sin(\theta_r+2\pi/3) & \cos(\theta_r+2\pi/3)\end{bmatrix}$	$K_{2r}^{3s}=\sqrt{\frac{2}{3}}\cdot\begin{bmatrix}\cos(\theta_r) & \sin(\theta_r)\\ \cos(\theta_r-2\pi/3) & \sin(\theta_r-2\pi/3)\\ \cos(\theta_r+2\pi/3) & \sin(\theta_r+2\pi/3)\end{bmatrix}$	
Voltage & Torque Equations in α-β frame of references		
$\begin{bmatrix}v_\alpha\\ v_\beta\end{bmatrix}=\begin{bmatrix}r_s & 0\\ 0 & r_s\end{bmatrix}\begin{bmatrix}i_\alpha\\ i_\beta\end{bmatrix}+\begin{bmatrix}L_{\alpha\beta} & 0\\ 0 & L_{\alpha\beta}\end{bmatrix}\begin{bmatrix}pi_\alpha\\ pi_\beta\end{bmatrix}+\lambda_m\,\omega_r\begin{bmatrix}-\sin(\theta_r)\\ \cos(\theta_r)\end{bmatrix}$	$\begin{bmatrix}v_\alpha\\ v_\beta\end{bmatrix}=\begin{bmatrix}r_s & 0\\ 0 & r_s\end{bmatrix}\begin{bmatrix}i_\alpha\\ i_\beta\end{bmatrix}+\begin{bmatrix}L_{\alpha\beta} & 0\\ 0 & L_{\alpha\beta}\end{bmatrix}\begin{bmatrix}pi_\alpha\\ pi_\beta\end{bmatrix}+\lambda_m\,\omega_r\begin{bmatrix}\cos(\theta_r)\\ \sin(\theta_r)\end{bmatrix}$	$\begin{bmatrix}v_\alpha\\ v_\beta\end{bmatrix}=\begin{bmatrix}r_s & 0\\ 0 & r_s\end{bmatrix}\begin{bmatrix}i_\alpha\\ i_\beta\end{bmatrix}+\begin{bmatrix}L_{\alpha\beta} & 0\\ 0 & L_{\alpha\beta}\end{bmatrix}\begin{bmatrix}pi_\alpha\\ pi_\beta\end{bmatrix}+\lambda_m\,\omega_r\begin{bmatrix}\cos\theta_r\\ -\sin\theta_r\end{bmatrix}$

$T_e(i_\alpha, i_\beta, \theta_r) = P.\lambda_m.\left(i_\beta \cos(\theta_r) - i_\alpha \sin(\theta_r)\right)$	$T_e(i_\alpha, i_\beta, \theta_r) = P.\lambda_m.\left(i_\alpha \cos(\theta_r) + i_\beta \sin(\theta_r)\right)$	$T_e(i_\alpha, i_\beta, \theta_r) = P.\lambda_m.\left(i_\alpha \cos(\theta_r) - i_\beta \sin(\theta_r)\right)$
Voltages Equation in q-d frame of references		
$\begin{bmatrix} v_q \\ v_d \end{bmatrix} = \begin{bmatrix} r_s & \omega_r L_{\alpha\beta} \\ -\omega_r L_{\alpha\beta} & r_s \end{bmatrix} . \begin{bmatrix} i_q \\ i_d \end{bmatrix} + \begin{bmatrix} L_{\alpha\beta} & 0 \\ 0 & L_{\alpha\beta} \end{bmatrix} . \begin{bmatrix} pi_q \\ pi_d \end{bmatrix} + \lambda_m \ \omega_r \begin{bmatrix} 1 \\ 0 \end{bmatrix}$		
Torque Equation in q-d frame of references		
$T_e(i_q) = P.\lambda_m.i_q$		